ETERNAL SOLUTIONS TO THE RICCI FLOW

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1. The result

We consider solutions to the Ricci flow equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$$

on a manifold X of dimension n. We say the solution is eternal if it is defined for all time $-\infty < t < \infty$. We are interested in solutions which are complete (which is a way of saying they are also defined for "all" of space) and which have their Riemannian curvature uniformly bounded for all space and time. This is a serious restriction; by the work of W. X. Shi [2] we know then that all the covariant derivatives of the curvature are bounded.

Examples of eternal solutions which are complete with bounded curvature are provided by solitons. These are solutions which move under a one-parameter family of diffeomorphisms. If this comes from exponentiating a vector field V_i, then we have a soliton when

$$D_i V_j + D_j V_i = 2R_{ij},$$

since the metric changes by its Lie derivative along the vector field. When the vector field is the gradient of a function we say we have a gradient soliton. If $V_i = D_i f$, the equation for a gradient soliton is

$$D_i D_j f = R_{ij},$$

so the Ricci tensor is the Hessian of a function. In dimensions 2 and 3 for sure, and probably in all higher dimensions too, there exists a complete gradient soliton with bounded curvature and strictly positive curvature operator which is rotationally symmetric around an origin; it can be found by solving an ODE.

Eternal solutions with bounded curvature are important because they occur as models for slowly forming singularities. Our main result is the following start at a classification.

1.1. Main theorem. *Any complete simply connected eternal solution to the Ricci flow with uniformly bounded curvature and strictly positive curvature operator where the scalar curvature R assumes its maximum is necessarily a gradient soliton.*

The proof comes from considering the Harnack inequality for the Ricci flow (see [1]) where we study a quadratic form which vanishes on solitons and is weakly positive on any solution. The condition that R assumes its maximum implies that this quadratic form has a large null space. We can then use the strong maximum principle to see that the only way this can happen is when we are on a soliton.

2. The Harnack inequality

In our paper [1] we prove a Harnack inequality for the Ricci flow. There we only assume the solution exists on $0 < t < T$, and we derive an estimate with terms $1/t$ in it. There is an interesting but simple procedure for getting rid of them when our solution is eternal. If we have a solution on $\alpha < t < T$, we can replace t by $t - \alpha$ in the Harnack inequality. Then if $\alpha \to -\infty$, the expression $1/(t - \alpha) \to 0$ and disappears! That proves the following result.

2.1. Theorem. *Suppose we have a complete eternal solution to the Ricci flow with uniformly bounded curvature and nonnegative curvature operator. Let*

$$M_{ab} = \Delta R_{ab} - \tfrac{1}{2} D_a D_b R + 2 R_{acbd} R_{cd} - R_{ac} R_{bc}$$

and

$$P_{abc} = D_a R_{bc} - D_b R_{ac}$$

and consider the quadratic form

$$Z = M_{ab} W_a W_b + 2 P_{abc} V_{ab} W_c + R_{abcd} U_{ab} U_{cd}$$

where W_a is a one-form and U_{ab} is a two-form. Then Z is weakly positive, so $Z \geq 0$ for any choice of W and V.

(Note we have just dropped the term $\frac{1}{t} R_{ab}$ from the definition of M_{ab} in [1].)

We shall also need the following computations, which are the basis of the proof of the Harnack inequality. They come from [1] by dropping all terms with $1/t$.

2.2. Computation. At a point where

$$(D_t - \Delta) W_a = 0, \qquad (D_t - \Delta) U_{ab} = 0,$$
$$D_a W_b = 0 \quad \text{and} \quad D_a V_{bc} = \tfrac{1}{2}(R_{ab} W_c - R_{ac} W_b)$$

we have

$$(D_t - \Delta)Z = 2R_{acbd}M_{cd}W_aW_b - 2P_{acd}P_{bdc}W_aW_b$$
$$+ 8R_{adce}P_{dbe}U_{ab}W_c + 4R_{aecf}R_{bedf}U_{ab}U_{cd}$$
$$+ [P_{abc}W_c + R_{abcd}U_{cd}][P_{abe}W_eR_{abef}U_{ef}].$$

2.3. Computation. At a point where the quadratic form Z can be written as a sum of squares of linear forms

$$Z = \sum_M (X_a^M W_a + Y_{ab}^M U_{ab})^2,$$

the quadratic form

$$Q = 2R_{acbd}M_{cd}W_aW_b - 2P_{acd}P_{bdc}W_aW_b$$
$$+ 8R_{adce}P_{bde}U_{ab}W_c + 4R_{aecf}R_{bedf}U_{ab}U_{cd}$$

is given by

$$Q = \sum_{MN}(Y_{ac}^M X_c^N W_a - Y_{ac}^N X_c^M W_a - 2Y_{ac}^M Y_{bc}^N U_{ab})^2,$$

which is also a sum of squares of linear forms.

3. The idea

We now give the idea of the proof, depending on a lemma which we prove later. The quadratic form Z is defined on the space of $W \oplus U$ in $\wedge^1 \oplus \wedge^2$. If the curvature operator $R_{abcd}U_{ab}U_{cd}$ is strictly positive, then Z is strictly positive on the subspace $\wedge^2 \subseteq \wedge^1 \oplus \wedge^2$ where $W = 0$. Therefore its null space has dimension at most $\dim \wedge^1 = n$. If we write $U = V \wedge W$ so that

$$U_{ab} = \tfrac{1}{2}(V_aW_b - V_bW_a)$$

and sum over an orthonormal basis of W_a's

$$\sum_W Z(W \oplus V \wedge W) = \frac{\partial R}{\partial t} + 2D_aR \cdot V_a + R_{ab}V_aV_b$$

and by letting $V = 0$ we get

$$\sum_W Z(W \oplus 0) = \frac{\partial R}{\partial t}.$$

At a point where R assumes its maximum, we will have $\partial R/\partial t = 0$, and hence letting W run through a basis E_a we have $Z(E_a \oplus 0) \geq 0$ for each

a, so $Z(E_a \oplus 0) = 0$ for each a. Thus Z has a null space of dimension exactly n at this point.

We then want to argue from the strong maximum principle that at any earlier time the null space of Z has dimension n at every point. This is Lemma 4.1. We can then find a smooth tensor L_{abc} with $L_{abc} + L_{bac} = 0$ such that for any W_a if

$$U_{ab} = L_{abc} W_c$$

then $W \oplus U$ is the null space of Z. Since the null space of Z satisfies

$$M_{ac} W_a + P_{abc} U_{ab} = 0$$

and

$$P_{abc} W_c + R_{abcd} U_{cd} = 0$$

we find that

$$M_{cd} L_{abc} P_{abd} = 0 \quad \text{and} \quad P_{cde} + R_{abcd} L_{abe} = 0.$$

We can however do even better. Since the null space of Z cannot contract, we see from Computations 2 and 3 that the null space of Z must be contained in the null space of Q. Now $W \oplus U$ lies in the null space of Z precisely when

$$X_a^M W_a + Y_{ab}^M U_{ab} = 0$$

for all M, and this happens whenever $U_{ab} = L_{abc} W_c$, which shows us

$$X_c^M = -Y_{ab}^M L_{abc}.$$

But then $W \oplus U$ also lies in the null space of Q, so

$$Y_{ac}^M X_c^N W_a - Y_{ac}^N X_c^M W_a - 2 Y_{ac}^M Y_{bc}^N U_{ab} = 0$$

for all M and N. Writing U in terms of L and W and writing X in terms of L and Y and simplifying algebraically give

$$Y_{ec}^M Y_{ab}^N L_{abc} - Y_{ec}^N Y_{ab}^M L_{acb} + 2 Y_{ac}^M Y_{bc}^N L_{abe} = 0$$

for all M and N.

Now from the definition of X and Y

$$R_{abcd} = \sum_M Y_{ab}^M Y_{cd}^M$$

and since the curvature operator is strictly positive the Y_{ab}^M can be chosen as an orthogonal basis for the two-forms. In fact we can take

$$Y_{ab}^M = \lambda^M E_{ab}^M$$

where E^M_{ab} is an orthonormal basis for the two-forms, and $(\lambda^M)^2$ are the eigenvalues of the curvature operator. Then dividing the previous expression by $\lambda^M \lambda^N$ gives

$$E^M_{ec} E^N_{ab} L_{abc} - E^N_{ec} E^M_{ab} L_{abc} + 2E^M_{ac} E^N_{bc} L_{abe} = 0 \, .$$

We now multiply by $E^M_{pq} E^N_{rs}$ and sum over M and N, using

$$\sum_M E^M_{ab} E^M_{cd} = \tfrac{1}{2}(g_{ac} g_{bd} - g_{ad} g_{bc})$$

and simplifying to conclude that

$$g_{pe} L_{rsq} - g_{qe} L_{rsp} - g_{re} L_{pqs} - g_{se} L_{pqr}$$
$$+ g_{qs} L_{pre} - g_{qr} L_{pse} - g_{ps} L_{qre} + g_{pr} L_{qse} = 0$$

for any choice of p, q, r, s, and e.

3.1. Lemma. *There exists a V_a such that*

$$L_{abc} = \tfrac{1}{2}(V_a g_{bc} - V_b g_{ac}) \, .$$

Proof. This follows algebraically from the above relation, and is in fact equivalent to it. To see this, take the trace on q and s. This gives

$$(n-2)L_{pre} - L_{rep} - L_{epr} = X_p g_{re} - X_r g_{pe}$$

when $X_p = L_{pqq}$ is the trace. Now we cyclically permute p, r, and e and sum to conclude that

$$(n-4)(L_{pre} + L_{rep} + L_{epr}) = 0$$

and so

$$L_{pre} + L_{rep} + L_{epr} = 0$$

at least when $n \neq 4$. It is also true when $n = 4$, as we see differently. We only need to check it for p, r, e all distinct, so take $p = 1$, $q = 2$, $r = 3$, $s = e = 4$ in the original formula and get $L_{123} = 0$. Then surely $L_{123} + L_{231} + L_{312} = 0$ also, and this is good enough.

Now if we substitute this in the formula above for the trace, we get

$$(n-1)L_{pre} = X_p g_{re} - X_r g_{pe}$$

which gives

$$L_{pre} = \tfrac{1}{2}(V_p g_{re} - V_r g_{pe})$$

for $V_p = \frac{2}{n-1} X_p$. This proves the lemma. q.e.d.

The lemma produces our desired smooth vector field along which the solution is supposed to flow. Substituting back in our formulas for the null space of Z gives

$$M_{bc} + V_a P_{abc} = 0 \quad \text{and} \quad P_{abc} = R_{abcd} V_d \, .$$

Now if we differentiate the second expression and use the first and the relations

$$M_{bc} = D_a P_{abc} + R_{bdce} R_{de} \quad \text{and} \quad D_a R_{abcd} = P_{cdb}$$

(which hold on any Riemannian manifold) we find that everything simplifies to

$$R_{abcd}(D_b V_d - R_{bd}) = 0 \, .$$

Now we let

$$T_{ab} = D_a V_b - R_{ab}$$

and observe that $R_{abcd} T_{bd} = 0$.

3.2. Lemma. *The tensor T_{ab} is symmetric.*

Proof. Let $T_{ab} = S_{ab} + A_{ab}$ be its decomposition into symmetric and antisymmetric parts. Then

$$R_{abcd} T_{bd} = R_{abcd} S_{bd} + R_{abcd} A_{bd}$$

is also such a decomposition. Therefore $R_{abcd} A_{bd} = 0$. Using the Bianchi identity we find that this is equivalent to $R_{abcd} A_{cd} = 0$ and since the curvature operator is strictly positive we must have $A_{cd} = 0$, so T_{ab} is symmetric.

In dimension three we could already conclude that $T_{ab} = 0$ from $R_{abcd} T_{bd} = 0$. For in a basis where T_{ab} is diagonal we get the equations

$$R_{1212} T_{22} + R_{1313} T_{33} = 0,$$
$$R_{1212} T_{11} + R_{2323} T_{33} = 0,$$
$$R_{1313} T_{11} + R_{2323} T_{22} = 0$$

and the coefficient matrix has determinant

$$2 R_{1212} R_{1313} R_{2323} > 0$$

so the only solution is

$$T_{11} = T_{22} = T_{33} = 0 \, .$$

In higher dimensions this does not suffice, and we have to work harder. A lot harder.

First note that

$$D_a V_b = R_{ab} + T_{ab} = D_b V_a$$

so that when the manifold is simply connected we can solve globally for a function f with

$$D_a f = V_a.$$

We then have the equations

(1) $$R_{abcd} T_{bd} = 0,$$
(2) $$D_a D_b f = R_{ab} + T_{ab},$$
(3) $$P_{abc} = R_{abcd} D_d f,$$
(4) $$M_{bc} + D_a f \cdot P_{abc} = 0$$

and we now work from here. Differentiate (2) and switch derivatives and use (3) to show that

(5) $$D_a T_{bc} = D_b T_{ac}$$

so that T_{ab} is a Codazzi tensor. Now apply the operator $(D_t - \Delta)$ to (2) and let

(6) $$(D_t - \Delta) f = h$$

and use the commutation formula

$$(D_t - \Delta) D_a D_b f = D_a D_b (D_t - \Delta) f + 2 R_{acbd} D_c D_d f$$

and

$$(D_t - \Delta) R_{ab} = 2 R_{acbd} R_{cd}$$

to compute that

(7) $$(D_t - \Delta) T_{ab} = D_a D_b h.$$

If we differentiate (5) again to get

$$D_a D_b T_{cd} = D_a D_c T_{bd}$$

and switch D_a and D_b we have

$$D_a D_c T_{bd} - D_b D_a T_{cd} = R_{abce} T_{de} + R_{abde} T_{ce}$$

and then if we cyclically permute a, b, c and sum, and use the Bianchi identity on R_{abce}, we get

$$R_{abde} T_{ce} + R_{bcde} T_{ae} + R_{cade} T_{be} = 0.$$

If we then trace on a and d we get

(8) $$R_{be} T_{ce} = R_{ce} T_{be}$$

which shows R_{ab} and T_{ab} commute.

Going back to (7), by the commutator formula

$$(D_t - \Delta)D_a T_{bc} = D_a(D_t - \Delta)T_{bc} + 2R_{adbc}D_d T_{ec} + 2R_{adce}D_d T_{be}$$

and this gives

$$(D_t - \Delta)D_a T_{bc} = D_a D_b D_c h + 2R_{adbe}D_c T_{de} + 2R_{adce}D_b T_{de}.$$

Now switch a and b and subtract. We then have

$$R_{abcd}D_d h + 2R_{adce}D_b T_{de} - 2R_{bdce}D_a T_{de} = 0.$$

Going back and differentiating (1) gives

$$R_{adce}D_b T_{de} + T_{de}D_b R_{adce} = 0$$

and doing this on the other term also shows that

$$R_{abcd}D_d h = 2T_{de}(D_b R_{adce} - D_a R_{bdce})$$

and by the second Bianchi identity

(9) $$R_{abcd}D_d h = 2T_{de}D_d R_{abce}.$$

Now from [1] we recall the evolution formulas

$$(D_t - \Delta)R_{abcd} = 2R_{aebf}R_{cedf} - 2R_{aebf}R_{decf}$$
$$+ 2R_{aecf}R_{bedf} - 2R_{aedf}R_{becf},$$
$$(D_t - \Delta)P_{abc} = 2R_{adbe}P_{dec} + 2R_{adce}P_{dbe}$$
$$+ 2R_{bdce}P_{ade} - 2R_{de}D_d R_{abce},$$
$$(D_t - \Delta)M_{ab} = 2R_{acbd}M_{cd} + 2R_{cd}[D_c P_{dab} + D_c P_{dba}]$$
$$+ 2P_{acd}P_{bcd} - 4P_{acd}P_{bdc} + 2R_{cd}R_{ce}R_{adbe}.$$

When we apply the evolution operator $(D_t - \Delta)$ to formula (3) there are many cancellations, after which we just recover (9), so we obtain no new information. But if we apply $(D_t - \Delta)$ to (4) and use the given identities, then after many cancellations we are left with one new term! Namely we get

(10) $$R_{abcd}T_{be}T_{de} = 0$$

which looks just like (1) but is in fact much better since the matrix T_{ab} now is squared.

3.3. Lemma. *The tensor $T_{ab} = 0$.*

Proof. Choose a basis where T_{ab} is diagonal. Then

$$R_{abcd}T_{be}T_{de} = 0$$

implies that for $a = c = 1$

$$R_{1212}T_{22}^2 + R_{1313}T_{33}^2 + \cdots + R_{1n1n}T_{nn}^2 = 0$$

and since $R_{1212} > 0$, $R_{1313} > 0$, $\cdots$, $R_{1n1n} > 0$ we get $T_{22} = 0$, $T_{33} = 0$, $\cdots$, $T_{nn} = 0$. Similarly all the diagonal entries are zero. q.e.d.

We now have $D_a V_b = R_{ab}$, so the metric is in fact a Ricci soliton.

4. The strong maximum principle

We now prove the lemma we need to make the previous argument work.

4.1. Lemma. *If the quadratic form Z has null space of dimension strictly less than n at some point at $t = 0$, then it has rank strictly less than n at every point for any time $t > 0$.*

Proof. This will be a consequence of the usual strong maximum principle, which assures us that if we have a function $F \geq 0$ which solves

$$(D_t - \Delta)F = 0$$

for $t \geq 0$ and if we have $F > 0$ at some point when $t = 0$, then we have $F > 0$ everywhere as soon as $t > 0$.

Suppose now that Z has null space of dimension strictly less than n at some point X^0 at $t = 0$. By picking X^0 in general position we may assume the dimension of the null space is constant in a neighborhood of X^0. We can then choose a smooth vector field Y_a with support in this neighborhood so that $Y_a \neq 0$ at the point X^0 but Y_a is orthogonal to the null space of Z, in the sense that if Z vanishes on $W_a \oplus U_{bc}$ then $Y_a W_a = 0$. We then define a matrix by

$$F_{ab} = Y_a Y_b$$

at $t = 0$, and allow F_{ab} to evolve by the heat equation

$$(D_t - \Delta)F_{ab} = 0.$$

Since $F_{ab} \geq 0$ as a matrix at $t = 0$, it will remain so for $t \geq 0$ by the maximum principle. Let $F = F_{aa}$ be the trace. Then $F > 0$ at the point X^0 at $t = 0$, so $F > 0$ everywhere as soon as $t > 0$. But this means F_{ab} has rank at least one.

The quadratic form $F_{ab}W_a W_b$ vanishes on the null space of Z, so by multiplying F_{ab} by $\varepsilon > 0$ sufficiently small we can arrange things so that $Z \geq F_{ab}W_a W_b$ at $t = 0$. Now we claim that this inequality will be preserved for $t \geq 0$. Whereupon we are done; inasmuch as $R_{abcd}U_{ab}U_{cd} > 0$ for $U_{ab} \neq 0$, so if $W_a \oplus U_{bc}$ lies in the null space of Z we must have

$W_a \neq 0$, which shows the dimension of the null space of Z cannot exceed the dimension of the null space of F_{ab}.

Note that $R_{acbd}F_{cd} \geq 0$ since $F_{cd} \geq 0$ and R_{acbd} has positive sectional curvature. Therefore the following result will suffice.

4.2. Theorem. *Let F_{ab} be uniformly bounded with $F_{ab} \geq 0$ and satisfy*

$$(D_t - \Delta)F_{ab} \leq 2R_{acbd}F_{cd}.$$

Suppose $Z \geq F_{ab}W_aW_b$ when $t = 0$. Then this remains true for $t \geq 0$.

Proof. We modify the Harnack expression Z by letting

$$\widehat{Z} = \widehat{M}_{ab}W_aW_b + 2P_{abc}U_{ab}W_c + \widehat{R}_{abcd}U_{ab}U_{cd}$$

where we let

$$\widehat{M}_{ab} = M_{ab} - F_{ab} + \varphi g_{ab},$$
$$\widehat{R}_{abcd} = R_{abcd} + \tfrac{1}{2}\psi(g_{ac}g_{bd} - g_{ad}g_{bc})$$

where φ and ψ are functions we will choose later and then let go to zero. We compute the modifications to the evolution of $\widehat{Z}$ in Computation 2.2 from the introduction of F_{ab}, φ and ψ. This gives us the following result, using C as a constant to bound $|Rm|$, $|DRm|$, $|D^2Rm|$ and $|F_{ab}|$, and we assume $\psi \leq 1$.

4.3. Lemma. *We have*

$$
\begin{aligned}
(D_t - \Delta)\widehat{Z} = {} & 2\widehat{R}_{acbd}\widehat{M}_{cd}W_aW_b - 2P_{acd}P_{bdc}W_aW_b \\
& + 8\widehat{R}_{adce}P_{dbe}U_{ab}W_c + 4\widehat{R}_{aecf}\widehat{R}_{bedf}U_{ab}U_{cd} \\
& + [P_{abc}W_c + R_{abcd}U_{cd}][P_{abe}W_e + R_{abef}U_{ef}] \\
& - [(D_t - \Delta)F_{ab} - 2R_{acbd}F_{cd}]W_aW_b \\
& + [(D_t - \Delta)\varphi]|W|^2 + [D_t\psi]|U|^2 - E
\end{aligned}
$$

where the error E is bounded by

$$E \leq C(\varphi + \psi)|W|^2 + C\psi|U|^2$$

at a point where

$$(D_t - \Delta)W_a = 0, \qquad (D_t - \Delta)U_{ab} = 0,$$
$$D_aW_b = 0 \quad and \quad D_aU_{bc} = \tfrac{1}{2}(R_{ab}W_c - R_{ac}W_b).$$

Proof. Few new terms can occur since we do not have to worry about space derivatives falling on W_a or ψ. The only trick is to bound the cross-term

$$C\psi|U||W| \leq C\psi|U|^2 + C\psi|W|^2$$

which is obvious.

Now to make the argument work we need to have

$$(D_t - \Delta)\varphi > C(\varphi + \psi) \quad \text{and} \quad D_t\psi > C\psi.$$

This is easy to achieve. We take $\psi = \delta e^{At}$ with δ small and $A > C$. Then we take $\varphi = \delta e^{At}f(x)$ where f is the function constructed with $f(x) \to \infty$ as $x \to \infty$ but all the covariant derivatives of f bounded from Lemma 5.1 in [1], and A is large compared to C before and to C bounding Δf. We then have $\psi \geq \delta > 0$ and $\varphi \to \infty$ when $X \to \infty$. Since M_{ab}, F_{ab} and P_{abc} are bounded while R_{abcd} is positive, it is clear that $\hat{Z}$ is strictly positive outside of a compact set. If $\hat{Z}$ ever becomes zero, there will be a first time t^0 this happens, and a point X^0 and an eigenvector W_a^0 and U_{ab}^0 where $\hat{Z}$ is zero. Extend W_a and U_{ab} to sections with $W_a = W_a^0$ and $U_{ab} = U_{ab}^0$ at (X^0, t^0) and so that

$$(D_t - \Delta)W_a = 0, \qquad (D_t - \Delta)U_{bc} = 0,$$
$$D_a W_b = 0 \quad \text{and} \quad D_a U_{bc} = \tfrac{1}{2}(R_{ab}W_c - R_{ac}W_b)$$

at (X^0, t^0). Then Computation 2.3 shows us that $(D_t - \Delta)\hat{Z} > 0$ at (X^0, t^0). However $D_t\hat{Z} \leq 0$ and $\Delta\hat{Z} \geq 0$ there, so we have a contradiction. Therefore $\hat{Z}$ can never become zero. We now let $\delta \to 0$ in the choice of φ and ψ (note A remains the same) and we recover the theorem.

References

[1] R. S. Hamilton, *The Harnack estimate for the Ricci flow*, J. Differential Geometry **37** (1993) 225–243.

[2] W. X. Shi, *Ricci deformation of the metric on complete noncompact Riemannian manifolds*, J. Differential Geometry **30** (1989) 303–394.

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